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EXOTIC DEFORMATION QUANTIZATION

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1. Introduction

Let $\mathcal{A}$ be one of the following commutative associative algebras: the algebra of all smooth functions on the plane: $\mathcal{A} = C^\infty(\mathbf{R}^2)$, or the algebra of polynomials $\mathcal{A} = \mathbf{C}[p,q]$ over $\mathbf{R}$ or $\mathbf{C}$. There exists a non-trivial formal *associative* deformation of $\mathcal{A}$ called the *Moyal $\star$-product* (or the standard $\star$-product). It is defined as an associative operation $\mathcal{A}^{\otimes 2} \to \mathcal{A}[[\hbar]]$ where $\hbar$ is a formal variable. The explicit formula is:

$$F \star_\hbar G = FG + \sum_{k \geq 1} \frac{(i\hbar)^k}{2^k k!} \{F, G\}_k, \tag{1}$$

where $\{F,G\}_1 = \dfrac{\partial F}{\partial p}\dfrac{\partial G}{\partial q} - \dfrac{\partial F}{\partial q}\dfrac{\partial G}{\partial p}$ is the standard Poisson bracket, and the higher order terms are:

$$\{F,G\}_k = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} k \\ i \end{pmatrix} \frac{\partial^k F}{\partial p^{k-i}\partial q^i} \frac{\partial^k G}{\partial p^i \partial q^{k-i}}. \tag{2}$$

The Moyal product is the *unique* (modulo equivalence) non-trivial formal deformation of the associative algebra $\mathcal{A}$ (see [13]).

Definition 1. A formal associative deformation of $\mathcal{A}$ given by formula (1) is called a *$\star$-product* if the following hold:

1) the first order term coincides with the Poisson bracket: $\{F,G\}_1 = \{F,G\}$;

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2) the higher order terms $\{F,G\}_k$ are given by differential operators vanishing on constants: $\{1,G\}_k = \{F,1\}_k = 0$;

3) $\{F,G\}_k = (-1)^k\{G,F\}_k$.

Definition 2. Two $\star$-products $\star_\hbar$ and $\star'_\hbar$ on $\mathcal{A}$ are called *equivalent* if there exists a linear mapping $A_\hbar : \mathcal{A} \to \mathcal{A}[[\hbar]]$ such that

$$A_\hbar(F) = F + \sum_{k=1}^{\infty} A_k(F)\hbar^k$$

intertwining the operations $\star_\hbar$ and $\star'_\hbar$: $A_\hbar(F) \star'_\hbar A_\hbar(G) = A_\hbar(F \star_\hbar G)$.

Consider now $\mathcal{A}$ as a Lie algebra; the commutator is given by the Poisson bracket. The Lie algebra $\mathcal{A}$ has a unique (modulo equivalence) non-trivial formal deformation called the *Moyal bracket* or the Moyal $\star$-commutator: $\{F,G\}_t = \frac{1}{i\hbar}(F \star_\hbar G - G \star_\hbar F)$, where $t = -\hbar^2/2$.

The well-known De Wilde–Lecomte theorem [4] states the existence of a non-trivial $\star$-product for an arbitrary symplectic manifold. The theory of $\star$-products is a subject of *deformation quantization*. The geometrical proof of the existence theorem was given by B. Fedosov [9] (see [8] and [20] for clear explanation and survey of recent progress).

The main idea of this paper is to consider the algebra $\mathcal{F}(M)$ of functions (with singularities) on the cotangent bundle T^*M which are *Laurent polynomials* on the fibers. In contrast to the above algebra $\mathcal{A}$ it turns out that for such algebras the standard $\star$-product is no more unique at least if M is one-dimensional: $\dim M = 1$.

We consider $M = S^1, \mathbf{R}$ in the real case, and $M = \mathcal{H}$ (the upper half-plane) in the holomorphic case. The main result of this paper is an explicit construction of a new $\star$-product on the algebra $\mathcal{F}(M)$ non-equivalent to the standard Moyal product. This $\star$-product is equivariant with respect to the Möbius transformations. The construction is based on the bilinear SL_2-equivariant operations on tensor-densities on M, known as *Gordan transvectants* and *Rankin-Cohen brackets*.

We study the relations between the new $\star$-product and extensions of the Lie algebra $\mathrm{Vect}(S^1)$.

The results of this paper are closely related to those of the recent work of P. Cohen, Yu. Manin and D. Zagier [3] where a one-parameter family of associative products on the space of classical modular forms is constructed using the same SL_2-equivariant bilinear operations.

2. Definition of the exotic $\star$-product

2.1 Algebras of Laurent polynomials. Let $\mathcal{F}$ be one of the following associative algebras of functions:

$$\mathcal{F} = \mathbf{C}[p, 1/p] \otimes C^\infty(\mathbf{R}) \quad \text{or} \quad \mathcal{F} = \mathbf{C}[p, 1/p] \otimes \mathrm{Hol}(\mathcal{H})$$

This means, it consists of functions of the type:

$$F(p,q) = \sum_{i=-N}^{N} p^i f_i(q), \tag{3}$$

where $f_i(q) \in C^\infty(\mathbf{R})$ in the real case, or $f_i(q)$ are holomorphic functions on the upper half-plane $\mathcal{H}$ or $f_i \in C[q]$ (respectively).

We will also consider the algebra of polynomials: $\mathbf{C}[p, 1/p, q]$ (Laurent polynomials in p) .

2.2 Transvectants. Consider the following bilinear operators on functions of one variable:

$$J_k^{m,n}(f,g) = \sum_{i+j=k} (-1)^i \binom{k}{i} \frac{(2m-i)!(2n-j)!}{(2m-k)!(2n-k)!} f^{(i)} g^{(j)}, \tag{4}$$

where $f = f(z), g = g(z),\ f^{(i)}(z) = \dfrac{d^i f(z)}{dz^i}$.

These operators satisfy a remarkable property: they are equivariant under Möbius (linear-fractional) transformations. Namely, suppose that the transformation $z \mapsto \frac{az+b}{cz+d}$ (with $ad - bc = 1$) acts on the arguments as follows:

$$f(z) \mapsto f\left(\frac{az+b}{cz+d}\right)(cz+d)^{2m}, \quad g(z) \mapsto g\left(\frac{az+b}{cz+d}\right)(cz+d)^{2n},$$

then $J_k^{m,n}(f,g)$ transforms as:

$$J_k^{m,n}(f,g)(z) \mapsto J_k^{m,n}(f,g)(\frac{az+b}{cz+d})(cz+d)^{2(m+n-k)}.$$

In other words, the operations (4) are bilinear SL_2-equivariant mappings on tensor-densities:

$$J_k^{mn} : \mathcal{F}_m \otimes \mathcal{F}_n \to \mathcal{F}_{m+n-k},$$

where $\mathcal{F}_l$ is the space of tensor-densities of degree $-l$: $\phi = \phi(z)(dz)^{-l}$.

The operations (4) were discovered more than one hundred years ago by Gordan [11] who called them the *transvectants*. They have been rediscovered many times: in the theory of modular functions by Rankin [18] and by Cohen [2] (so-called Rankin-Cohen brackets), in differential projective geometry by Janson and Peetre [12]. The "multi-dimensional transvectants" were defined in [14] in the context of the the Virasoro algebra and symplectic and contact geometry.

2.3 Main definition. Define the following bilinear mapping $\mathcal{F}^{\otimes 2} \to \mathcal{F}[[\hbar]]$, for $F = p^m f(q), G = p^n g(q)$, where $m, n \in \mathbf{Z}$, by putting:

$$F \widetilde{\star}_\hbar G = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{2^{2k}} p^{(m+n-k)} J_k^{m,n}(f,g), \tag{5}$$

Note, that the first order term coincides with the Poisson bracket.

This operation will be the main subject of this paper. We call it the *exotic $\star$-product.*

2.4 Remark. Another one-parameter family of operations on modular forms: $f \star^\kappa g = \sum_{n=0}^{\infty} t_n^\kappa(k,l) J_n^{kl}(f,g)$, where f and g are modular forms of weight k and l respectively, and $t_n^\kappa(k,l)$ are very interesting and complicated coefficients, is defined in [3].

3. Main theorems

We formulate here the main results of this paper. All the proofs will be given in Sections 4-7.

3.1 Non-equivalence. The Moyal $\star$-product (1) defines a nontrivial formal deformation of $\mathcal{F}$. We will show that the formula (5) defines a $\star$-product non-equivalent to the standard Moyal product.

Theorem 1. *The operation (5) is associative; it defines a formal deformation of the algebra $\mathcal{F}$ which is not equivalent to the Moyal product.*

The associativity of the product (5) is a trivial corollary of Proposition 1 below. To prove the non-equivalence, we will use the relations with extensions of the Lie algebra of vector fields on S^1: $\mathrm{Vect}(S^1) \subset \mathcal{F}$ (cf. Sec.5).

It is interesting to note that the constructed $\star$-product is equivalent to the standard Moyal product if we consider it on the algebra $C^\infty(T^*M \setminus M)$ of all smooth functions (not only Laurent polynomials on fibers); cf. Corollary 1 below.

3.2 sl_2-equivariance. The Lie algebra $sl_2(\mathbf{R})$ has two natural embeddings into the Poisson Lie algebra on $\mathbf{R}^2$: the *symplectic Lie algebra* $sp_2(\mathbf{R}) \cong sl_2(\mathbf{R})$ generated by quadratic polynomials (p^2, pq, q^2) and another one with generators: (p, pq, pq^2) which is called the *Möbius algebra.*

It is well-known that the Moyal product (1) is the unique non-trivial formal deformation of the associative algebra of functions on $\mathbf{R}^2$ equivariant under the action of the symplectic algebra. This means, (1) satisfies the Leibnitz property:

$$\{F, G \star_\hbar H\} = \{F, G\} \star_\hbar H + G \star_\hbar \{F, H\}, \tag{6}$$

where F is a quadratic polynomial (note that $\{F, G\}_t = \{F, G\}$ if F is a quadratic polynomial).

Theorem 2. *The product (5) is the unique formal deformation of the associative algebra $\mathcal{F}$ equivariant under the action of the Möbius algebra.*

The product (5) is the unique non-trivial formal deformation of $\mathcal{F}$ satisfying (6) for F from the Möbius sl_2 algebra.

3.3 Symplectomorphism Φ. The relation between the Moyal product and the product (5) is as follows. Consider the symplectic mapping

$$\Phi(p, q) = \left(\frac{p^2}{2}, \frac{q}{p}\right). \tag{7}$$

defined on $\mathbf{R}^2 \setminus \mathbf{R}$ in the real case and on $\mathcal{H}$ in the complex case.

Proposition 1. *The product (5) is the Φ-conjugation of the Moyal product:*

$$F \widetilde{\star}_\hbar G = F \star_\hbar^\Phi G := (F \circ \Phi \star_\hbar G \circ \Phi) \circ \Phi^{-1}. \tag{8}$$

Remark. The mapping (7) (in the complex case) can be interpreted as follows. It transforms the space of *holomorphic tensor-densities* of degree $-k$ on $\mathbf{CP}^1$ to the space $\mathbf{C}^k[p,q]$ of polynomials of degree k. Indeed, there exists a natural isomorphism $z^n(dz)^{-m} \mapsto p^m q^n$ (where $m \geq 2n$) and $(p^m q^n) \circ \Phi = p^{2m-n} q^n$.

3.4 Operator formalism. The Moyal product is related to the following Weil quantization procedure. Define the following differential operators:

$$\begin{aligned} \widehat{p} &= i\hbar \tfrac{\partial}{\partial q}, \\ \widehat{q} &= q \end{aligned} \tag{9}$$

satisfying the canonical relation: $[\widehat{p}, \widehat{q}] = i\hbar \mathbf{I}$. Associate to each polynomial $F = F(p,q)$ the differential operator $\widehat{F} = \mathrm{Sym} F(\widehat{p}, \widehat{q})$ symmetric in $\widehat{p}$ and $\widehat{q}$. The Moyal product on the algebra of polynomials coincides with the product of differential operators: $\widehat{F \star_\hbar G} = \widehat{F}\widehat{G}$.

We will show that the $\star$-product (5) leads to the operators:

$$\begin{aligned} \widehat{p}^{\Phi} &= (\tfrac{i\hbar}{2})^2 \Delta, \\ \widehat{q}^{\Phi} &= \tfrac{1}{4i\hbar}(\Delta^{-1} \circ A + A \circ \Delta^{-1}) \end{aligned} \tag{10}$$

(where $\Delta = \frac{\partial^2}{\partial q^2}$ and $A = 2q\frac{\partial}{\partial q} + 1$ is the dilation operator) also satisfying the canonical relation.

Remark that $\widehat{p}^{\Phi}$ and $\widehat{q}^{\Phi}$ given by (10) on the Hilbert space $L_2(\mathbf{R})$ are not equivalent to the operators (9) since $\widehat{q}^{\Phi}$ is symmetric but not self-adjoint (see [6] on this subject).

3.5 "Symplectomorphic" deformations. Let us consider the general situation.

Proposition 2. *Given a symplectic manifold V endowed with a $\star$-product $\star_\hbar$ and a symplectomorphism Ψ of V, if there exists a hamiltonian isotopy of Ψ to the identity, then the Ψ-conjugate product $\star_\hbar^\Psi$ defined according to the formula (8) is equivalent to $\star_\hbar$.*

Corollary 1. *The $\star$-product (5) considered on the algebra of all smooth functions $C^\infty(T^*\mathbf{R} \setminus \mathbf{R})$ is equivalent to the Moyal product.*

4. Möbius-invariance

In this section we prove Theorem 2. We show that the operations of transvectant (4) are Φ-conjugate of the terms of the Moyal product.

4.1 Lie algebra $\mathrm{Vect}(\mathbf{R})$ **and modules of tensor-densities**. Let $\mathrm{Vect}(\mathbf{R})$ be the Lie algebra of smooth (or polynomial) vector fields on $\mathbf{R}$:

$$X = X(x)\frac{d}{dx}$$

with the commutator

$$[X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}] = (X(x)Y'(x) - X'(x)Y(x))\frac{d}{dx}.$$

The natural embedding of the Lie algebra $sl_2 \subset \mathrm{Vect}(\mathbf{R})$ is generated by the vector fields $d/dx, xd/dx, x^2d/dx$.

Define a 1-parameter family of $\mathrm{Vect}(\mathbf{R})$-actions on $C^\infty(\mathbf{R})$ given by

$$L_X^{(\lambda)} f = X(x)f'(x) - \lambda X'(x)f(x), \tag{11}$$

where $\lambda \in \mathbf{R}$. Geometrically, $L_X^{(\lambda)}$ is the operator of Lie derivative on *tensor-densities* of degree $-\lambda$:

$$f = f(x)(dx)^{-\lambda}.$$

Denote $\mathcal{F}_\lambda$ the $\mathrm{Vect}(\mathbf{R})$-module structure on $C^\infty(\mathbf{R})$ given by (11).

4.2 Transvectant as a bilinear sl_2**-equivariant operator**. The operations (4) can be defined as bilinear mappings on $C^\infty(\mathbf{R})$ which are sl_2*-equivariant*:

Statement 4.1. For each $k = 0, 1, 2, \ldots$ there exists a unique (up to a constant) bilinear sl_2-equivariant mapping

$$\mathcal{F}_\mu \otimes \mathcal{F}_\nu \to \mathcal{F}_{\mu+\nu-k}.$$

It is given by $f \otimes g \mapsto J_k^{\mu,\nu}(f, g)$.

Proof. Straightforward (cf. [11], [12]).

4.3 Algebra $\mathcal{F}$ **as a module over** $\mathrm{Vect}(\mathbf{R})$. The Lie algebra $\mathrm{Vect}(\mathbf{R})$ can be considered as a Lie subalgebra of $\mathcal{F}$. The embedding $\mathrm{Vect}(\mathbf{R}) \subset \mathcal{F}$ is given by:

$$X(x)\frac{d}{dx} \mapsto pX(q).$$

The algebra $\mathcal{F}$ is therefore, a $\mathrm{Vect}(\mathbf{R})$-module.

Lemma 4.2. *The algebra $\mathcal{F}$ is decomposed to a direct sum of* Vect$(\mathbf{R})$*-modules:*

$$\mathcal{F} = \oplus_{m\in\mathbf{Z}}\mathcal{F}_m.$$

Proof. Consider the subspace of $\mathcal{F}$ consisting of functions homogeneous of degree m in p: $F = p^m f(q)$. This subspace is a Vect$(\mathbf{R})$-module isomorphic to $\mathcal{F}_m$. Indeed, $\{pX(q), p^m f(q)\} = p^m(Xf' - mX'f) = p^m L_X^{(m)} f$.

4.4 Projective property of the diffeomorphism Φ. The transvectants (4) coincide with the Φ-conjugate operators (2) from the Moyal product:

Proposition 4.3. *Let $F = p^m f(q), G = p^n g(q)$. Then*

$$\Phi^{*-1}\{\Phi^*F, \Phi^*G\}_k = \frac{k!}{2^k} p^{m+n-k} J_k^{m,n}(f,g). \tag{12}$$

Proof. The symplectomorphism Φ of $\mathbf{R}^2$ intertwines the symplectic algebra $sp_2 \equiv sl_2$ and the Möbius algebra: $\Phi^*(p, pq, pq^2) = (\frac{1}{2}p^2, \frac{1}{2}pq, \frac{1}{2}q^2)$. Therefore, the operation $\Phi^{*-1}\{\Phi^*F, \Phi^*G\}_k$ is Möbius-equivariant.

On the other hand, one has: $\Phi^*F = \frac{1}{2^m}p^{2m}f(\frac{q}{p})$ and $\Phi^*G = \frac{1}{2^n}p^{2m}g(\frac{q}{p})$. Since Φ^*F and Φ^*G are homogeneous of degree $2m$ and $2n$ (respectively), the function $\{\Phi^*F, \Phi^*G\}_k$ is also homogeneous of degree $2(m + n - k)$. Thus, the operation $\{F, G\}_k^\Phi = \Phi^{*-1}\{\Phi^*F, \Phi^*G\}_k$ defines a bilinear mapping on the space of tensor-densities $\mathcal{F}_m \otimes \mathcal{F}_n \to \mathcal{F}_{m+n-k}$ which is sl_2-equivariant.

Statement 4.1 implies that it is proportional to $J_k^{m,n}$. One easily verifies the coefficient of proportionality for $F = p^m, G = p^n q^k$, to obtain the formula (12).

Proposition 4.3 is proven.

Remark. Proposition 4.3 was proven in [15]. We do not know whether this elementary fact has been mentioned by classics.

4.5 Proof of Theorem 2. Proposition 4.3 implies that the formula (5) is a Φ-conjugation of the Moyal product and is given by the formula (8).

Proposition 1 is proven.

It follows that (5) is a $\star$-product on $\mathcal{F}$ equivariant under the action of the Möbius sl_2 algebra. Moreover, it is the unique $\star$-product with this property since the Moyal product is the unique $\star$-product equivariant under the action of the symplectic algebra.

Theorem 2 is proven.

5. Relation with extensions of the Lie algebra $\mathrm{Vect}(S^1)$

We prove here that the $\star$-product (5) is not equivalent to the Moyal product.

Let $\mathrm{Vect}(S^1)$ be the Lie algebra of vector fields on the circle. Consider the embedding $\mathrm{Vect}(S^1) \subset \mathcal{F}$ given by functions on $\mathbf{R}^2$ of the type: $X = pX(q)$ where $X(q)$ is periodical: $X(q+1) = X(q)$.

5.1 An idea of the proof of Theorem 1. Consider the formal deformations of the *Lie algebra* $\mathcal{F}$ associated to the $\star$-products (1) and (5). The restriction of the Moyal bracket to $\mathrm{Vect}(S^1)$ is identically zero. We show that the restriction of the $\star$-commutator

$$\widetilde{\{F,G\}}_t = \frac{1}{i\hbar}(F\widetilde{\star}_\hbar G - G\widetilde{\star}_\hbar F), \qquad t = -\frac{\hbar^2}{2}$$

associated to the $\star$-product (5) defines a series of non-trivial extensions of the Lie algebra $\mathrm{Vect}(S^1)$ by the modules $\mathcal{F}_k(S^1)$ of tensor-densities on S^1 of degree $-k$.

5.2 Extensions and the cohomology group $H^2(\mathrm{Vect}(S^1); \mathcal{F}_\lambda)$. Recall that *an extension* of a Lie algebra by its module is defined by a 2-cocycle on it with values in this module. To define an extension of $\mathrm{Vect}(S^1)$ by the module $\mathcal{F}_\lambda$ one needs therefore a bilinear mapping $c : \mathrm{Vect}(S^1)^{\otimes 2} \to \mathcal{F}_\lambda$ which satisfies the identity $\delta c = 0$:

$$c(X,[Y,Z]) + L_X^{(\lambda)} c(Y,Z) + (cycle_{X,Y,Z}) = 0.$$

(See [10]).

The cohomology group $H^2(\mathrm{Vect}(S^1); \mathcal{F}_\lambda)$ were calculated in [19] (see [10]). This group is trivial for each value of λ except $\lambda = 0, -1, -2, -5, -7$. The explicit formulæ for the corresponding non-trivial cocycles are given in [17]. If $\lambda = -5, -7$, then $\dim H^2(\mathrm{Vect}(S^1); \mathcal{F}_\lambda) = 1$, the cohomology group is generated by the unique (up to equivalence) non-trivial cocycle. We will obtain these cocycles from the $\star$-commutator.

5.3 Non-trivial cocycles on $\mathrm{Vect}(S^1)$.

Consider the restriction of the $\star$-commutator $\widetilde{\{\ ,\ \}}_t$ (corresponding to the $\star$-product (5)) to $\mathrm{Vect}(S^1) \subset \mathcal{F}$: let

$$X = pX(q), \;\; Y = pY(q);$$

then from (5) we have

$$\widetilde{\{X,Y\}}_t = \{X,Y\} + \sum_{k=1}^{\infty} \frac{t^k}{2^{2k+1}} \, \frac{1}{p^{2k-1}} \, J^{1,1}_{2k+1}(X,Y).$$

It follows from the Jacobi identity that the first non-zero term of the series $\widetilde{\{X,Y\}}_t$ is a 2-cocycle on $\mathrm{Vect}(S^1)$ with values in one of the $\mathrm{Vect}(S^1)$-modules $\mathcal{F}_k(S^1)$.

Denote for simplicity $J^{1,1}_{2k+1}$ by J_{2k+1}.

From the general formula (4) one obtains:

Lemma 5.1. *First two terms of* $\widetilde{\{X,Y\}}_t$ *are identically zero:* $J_3(X,Y) = 0, J_5(X,Y) = 0$*, the next two terms are proportional to:*

$$\text{(13)} \qquad \begin{array}{rl} J_7(X,Y) = & X'''Y^{(IV)} - X^{(IV)}Y''', \\ J_9(X,Y) = & 2(X'''Y^{(VI)} - X^{(VI)}Y''') \\ & -9(X^{(IV)}Y^{(V)} - X^{(V)}Y^{(IV)}). \end{array}$$

The transvectant J_7 defines therefore a 2-cocycle. It is a remarkable fact that the same fact is true for J_9:

Lemma 5.2. (See [17]). *The mappings*

$$J_7 : \mathrm{Vect}(S^1)^{\otimes 2} \to \mathcal{F}_{-5} \quad \textit{and} \quad J_9 : \mathrm{Vect}(S^1)^{\otimes 2} \to \mathcal{F}_{-7}$$

are 2-cocycles on $\mathrm{Vect}(S^1)$ *representing the unique non-trivial classes of the cohomology groups* $H^2(\mathrm{Vect}(S^1); \mathcal{F}_{-5})$ *and* $H^2(\mathrm{Vect}(S^1); \mathcal{F}_{-7})$ *respectively.*

Proof. Let us prove that J_9 is a 2-cocycle on $\mathrm{Vect}(S^1)$. The Jacobi identity for the bracket $\widetilde{\{\ ,\ \}}_t$ implies:

$$\{X, J_9(Y,Z)\} + J_9(X, \{Y,Z\}) + J_3(X, J_7(Y,Z)) \;\; + (cycle_{X,Y,Z}) = 0$$

for any $X = pX(q), Y = pY(q), Z = pZ(q)$. One checks that the expression $J_3(X, J_7(Y,Z))$ is proportional to $X'''(Y'''Z^{(IV)} - Y^{(IV)}Z''')$, so that

$$J_3(X, J_7(Y,Z)) \;\; + (cycle_{X,Y,Z}) = 0.$$

We obtain the following relation:

$$\{X, J_9(Y,Z)\} + J_9(X, \{Y,Z\}) \quad + (cycle_{X,Y,Z}) = 0,$$

which means that J_9 is a 2-cocycle. Indeed, recall that for any tensor density a, $\{pX, p^m a\} = p^m L^{(m)}_{X\,d/dx}(a)$. Thus, the last relation coincides with the relation $\delta J_9 = 0$.

Let us now show that the cocycle J_7 on $\mathrm{Vect}(S^1)$ is not trivial. Consider a linear differential operator $A : \mathrm{Vect}(S^1) \to \mathcal{F}_5$, given by: $A(X(q)d/dq) = (\sum_{i=0}^{K} a_i X^{(i)}(q))(dq)^5$. Then $\delta A(X,Y) = L^{(5)}_X A(Y) - L^{(5)}_Y A(X) - A([X,Y])$. The higher order part of this expression has a non-zero term $(5-K)a_K X' Y^{(K)}$ and therefore $J_7 \neq \delta A$.

In the same way one proves that the cocycle J_9 on $\mathrm{Vect}(S^1)$ is non-trivial.

Lemma 5.2 is proven.

It follows that the $\star$-product (5) on the algebra $\mathcal{F}$ is not equivalent to the Moyal product.

Theorem 1 is proven.

6. Operator representation

We are looking for an linear mapping (depending on $\hbar$) $F \mapsto \widehat{F}^\Phi$ of the associative algebra of Laurent polynomials $\mathcal{F} = \mathbf{C}[p, 1/p, q]$ into the algebra of formal pseudodifferential operators on $\mathbf{R}$ such that

$$\widehat{F \tilde{\star}_\hbar G}^{\Phi} = \widehat{F}^\Phi \widehat{G}^\Phi.$$

Recall that the algebra of Laurent polynomials $\mathbf{C}[p, 1/p, q]$ with the Moyal product is isomorphic to the associative algebra of pseudodifferential operators on $\mathbf{R}$ with polynomial coefficients (see [1]). This isomorphism is defined on the generators $p \mapsto \widehat{p}$, $q \mapsto \widehat{q}$ by the operators (9) and $p^{-1} \mapsto \widehat{p}^{-1}$:

$$\widehat{p}^{-1} = \frac{1}{i\hbar}(\partial/\partial q)^{-1}.$$

6.1 Definition. Put:

$$\widehat{F}^\Phi = \widehat{\Phi^* F}. \tag{14}$$

Then $\widehat{F}^\Phi \widehat{G}^\Phi = \widehat{\Phi^* F}\widehat{\Phi^* G} = \Phi^* F \star_\hbar \Phi^* G = \Phi^*(F \tilde{\star}_\hbar G) = \widehat{F \tilde{\star}_\hbar G}^{\Phi}$.

One obtains the formulæ (10). Indeed,

$$\widehat{p}^{\Phi} = \widehat{p^2/2} = \frac{(i\hbar)^2}{2}\frac{\partial^2}{\partial q^2}.$$

Since $q = \frac{1}{2}\left((\frac{1}{p})\widetilde{\star}_{\hbar}pq + pq\widetilde{\star}_{\hbar}(\frac{1}{p})\right)$, one gets:

$$\widehat{q}^{\Phi} = \frac{1}{4i\hbar}(\Delta \circ A + A \circ \Delta).$$

6.2 sl_2-equivariance. For the Möbius sl_2 algebra one has:

$$\begin{aligned} \widehat{p}^{\Phi} &= \frac{(i\hbar)^2}{2}\frac{\partial^2}{\partial q^2}, \\ \widehat{pq}^{\Phi} &= \frac{i\hbar}{4} + \frac{i\hbar}{2}q\frac{\partial}{\partial q}, \\ \widehat{pq^2}^{\Phi} &= \frac{q^2}{2}. \end{aligned}$$

Lemma 6.1. *The mapping $F \mapsto \widehat{F}^{\Phi}$ satisfies the Möbius-equivariance condition:*

$$\widehat{\{X, F\}}^{\Phi} = [\widehat{X}^{\Phi}, \widehat{F}^{\Phi}]$$

for $X \in sl_2$.

Proof. It follows immediately from Theorem 2. Indeed, the $\star$-product (5) is sl_2-equivariant (that is, satisfying the relation: $\{X, F\}_t = \{X, F\}$ for $X \in sl_2$).

Remark. Beautiful explicit formulæ for sl_2-equivariant mappings from the space of tensor-densities to the space of pseudodifferential operators are given in [3].

7. Hamiltonian isotopy

The simple calculations below are quite standard for the cohomological technique. We need them to prove Corollary 1 of Sec. 3.

Given a syplectomorphism Ψ of a symplectic manifold V and a formal deformation $\{\ ,\ \}_t$ of the Poisson bracket on V, we prove that if Ψ is isotopic to the identity, then the formal deformation $\{\ ,\ \}_t^{\Psi}$ defined by:

$$\{F, G\}_t^{\Psi} = \Psi^{*-1}\{\Psi^* F, \Psi^* G\}_t$$

is equivalent to $\{\ ,\ \}_t$. The similar proof is valid in the case of $\star$-products.

Recall that two symplectomorphisms Ψ and Ψ' of a symplectic manifold V are *isotopic* if there exists a family of functions $H_{(s)}$ on V such that the symplectomorphism $\Psi_1 \circ \Psi_2^{-1}$ is the flow of the Hamiltionian vector field with the Hamiltonian function $H_{(s)}, 0 \le s \le 1$.

Let $\Psi_{(s)}$ be the the flow of a family of functions $H = H_{(s)}$. We will prove that the equivalence class of the formal deformation $\{\ ,\ \}_t^{\Psi_{(s)}}$ does not depend on s.

7.1 Equivalence of homotopic cocycles. Let us first show that the cohomology class of the cocycle $C_3^{\Psi_{(s)}}$:

$$C_3^{\Psi_{(s)}}(F,G) = \Psi_{(s)}^{*-1} C_3(\Psi_{(s)}^* F, \Psi_{(s)}^* G)$$

does not depend on s. To do this, it is sufficient to prove that the derivative $\dot{C}_3 = \frac{d}{ds} C_3^{\Psi_{(s)}}|_{s=0}$ is a coboundary. One has

$$\dot{C}_3(F,G) = C_3(\{H,F\},G) + C_3(F,\{H,G\}) - \{H, C_3(F,G)\}.$$

The relation $\delta C_3 = 0$ implies:

$$\dot{C}_3(F,G) = \{F, C_3(G,H)\} - \{G, C_3(F,H)\} - C_3(\{F,G\},H).$$

This means, $\frac{d}{ds} C_3^{\Psi_{(s)}}|_{s=0} = \delta B_H$, where $B_H(F) = C_3(F,H)$.

7.2 General case. Let us apply the same arguments to prove that the deformations $\{\ ,\ \}_t^{\Psi_{(s)}}$ are equivalent to each other for all values of s. For this purpose we must show that there exists a family of mappings $A_{(s)}(F) = F + \sum_{k=1}^{\infty} A_{(s)_k}(F) t^k$ such that $A_{(s)}^{-1}(\{A_{(s)}(F), A_{(s)}(G)\}_t) = \{F,G\}_t$.

It is sufficient to verify the existence of a mapping $a(F) = \sum_{k=1}^{\infty} a_k(F) t^k$ (the derivative: $a(F) = d/ds(A_{(s)}(F))|_{s=s_0}$) such that

$$\frac{d}{ds}\{F,G\}_t^{\Psi_{(s)}}|_{s=s_0} = \{a(F),G\}_t + \{F, a(G)\} - a(\{F,G\}_t$$

Since

$$\frac{d}{ds}\{F,G\}_t^{\Psi_{(s)}}|_{s=s_0} = \{\{F,H\},G\}_t + \{F,\{G,H\}\}_t - \{\{F,G\}_t, H\},$$

from the Jacobi identity:

$$\{\{F,H\}_t,G\}_t + \{F,\{G,H\}_t\}_t - \{\{F,G\}_t,H\}_t = 0$$

one obtains that the mapping $a(F)$ can be written in the form:

$$a(F) = \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} C_{2k+1}(F, H_{(s_0)}) t^k.$$

7.3 Proof of Corollary 2. Consider the $\star$-product (8) given by $F \star_\hbar^\Phi G$, where $F \star_\hbar G$ is the Moyal product (1), and $\Phi : (p,q) \mapsto (p^2/2, q/p)$. It is defined on $\mathbf{R}^2 \setminus \mathbf{R}$.

The $\star$-product (8) on the algebra $C^\infty(\mathbf{R}^2 \setminus \mathbf{R})$ is equivalent to the Moyal product. Indeed, the symplectomorphism Φ is isotopic to the identity in the group of all smooth symplectomorphisms of $\mathbf{R}^2 \setminus \mathbf{R}$. The isotopy is: $\Phi_s : (p,q) \mapsto (\frac{p^{1+s}}{1+s}, \frac{q}{p^s})$, where $s \in [0,1]$.

Recall that the $\star$-product $F \star_\hbar^\Phi G$ on the algebra $\mathcal{F}$ is not equivalent to the Moyal product since it coincides with the product (5).

The family Φ_s does not preserve the algebra $\mathcal{F}$. Theorem 1 implies that F is not isotopic to the identity in the group of symplectomorphisms of $\mathbf{R}^2 \setminus \mathbf{R}$ preserving the algebra $\mathcal{F}$.

8. Discussion

8.1 Difficulties in multi-dimensional case.

There exist multi-dimensional analogues of transvectants [14] and [16].

Consider the projective space $\mathbf{RP}^{2n+1}$ endowed with the standard contact structure (or an open domain of the complex projective space $\mathbf{CP}^{2n+1}$). There exists an unique bilinear differential operator of order k on tensor-densities equivariant with respect to the action of the group Sp_{2n} (see [14], [16]):

$$J_k^{\lambda,\mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu-\frac{k}{n+1}}, \tag{15}$$

where $\mathcal{F}_\lambda = \mathcal{F}_\lambda(\mathbf{P}^{2n+1})$ is the space of tensor-densities on $\mathbf{P}^{2n+1}$ of degree $-\lambda$:

$$f = f(x_1, \dots, x_{2n+1})(dx_1 \wedge \dots dx_{2n+1})^{-\lambda}.$$

The space of tensor-densities $\mathcal{F}_\lambda(\mathbf{RP}^{2n+1})$ is *isomorphic as a module over the group of contact diffeomorphisms* to the space of homogeneous functions on $\mathbf{R}^{2n+2}$, and the isomorphism is given by:

$$f \mapsto F(y_1, \ldots, y_{2n+2}) = y_{2n+2}^{-\lambda(n+1)} f(\frac{y_1}{y_{2n+2}}, \ldots, \frac{y_{2n+1}}{y_{2n+2}}).$$

Then the operations (15) are defined as the restrictions of the terms of the standard $\star$-product on $\mathbf{R}^{2n+2}$.

The same formula (5) defines a $\star$-product on the space of tensor-densities on $\mathbf{CP}^{2n+1}$) (cf. [16]). However, there is no analogues of the symplectomorphism (7). I do not know if there exists a $\star$-product on the Poisson algebra $\mathbf{C}[y_{2n+2}, y_{2n+2}^{-1}] \otimes C^\infty(\mathbf{RP}^{2n+1})$ non-equivalent to the standard.

8.2 Classification problem. The classification (modulo equivalence) of $\star$-products on the Poisson algebra $\mathcal{F}$ is an interesting open problem. It is related to the calculation of cohomology groups $H^2(\mathcal{F};\mathcal{F})$ and $H^3(\mathcal{F};\mathcal{F})$. The following result was announced in [7]: dim $H^2(\mathcal{F};\mathcal{F}) = 2$.

Let us formulate a conjecture in the compact case. Consider the Poisson algebra $\mathcal{F}(S^1)$ of functions on $T^*S^1 \setminus S^1$ which are Laurent polynomials on the fiber: $F(p,q) = \sum_{-N \leq i \leq N} p^i f_i(q)$ where $f_i(q+1) = f_i(q)$.

Conjecture. *Every $\star$-product on $\mathcal{F}(S^1)$ is equivalent to (1) or (5).*

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